1.3. Random variable on the complex plane

If a researcher has several observations of a complex variable of economic indicators at his disposal, then they can be considered as random variables, since in the economy there are always a lot of random factors.

The complex random variable *Y* is the value

$$Y = y_r + iy_i , \quad (1.3.1)$$

in which y_r and y_i are real random variables, and *i* is an imaginary unit.

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All random processes that are studied by the economists have various features, which can mostly be reduced to some typical situations. These typical situations are well described in mathematical statistics and are called "the law of probability distribution." In order to understand which law of probability distribution a particular process belongs to, it is necessary to calculate its main characteristics and draw an appropriate conclusion from them.

The main characteristic of any random variable, including a complex random variable, is its mathematical expectation.

The mathematical expectation of a complex random variable (1.3.1) is called a complex number

$$m_Y = m_r + im_i . \quad (1.3.2),$$

where *mr* is the mathematical expectation of the real part, and *mi* is the mathematical expectation of the imaginary part of a complex random variable.

Since complex random numbers are points on the complex plane, the mathematical expectation of a complex random variable is also a point on the complex plane around which random complex variables are scattered. Moreover, for a normal distribution, such a rule is obvious - the probability that a random complex variable will be closer to its mathematical expectation is greater than the probability that it will be further from it.

The way the points are located on the complex plane is due to the presence or absence of a relationship between the real and imaginary parts of a complex random variable. Let us therefore consider two possible cases:

1) when both parts of a complex random variable are independent of each other and

2) when the real and imaginary parts of a complex random variable are interrelated.

The first case. The real and imaginary parts of a complex random variable are independent of each other. In modern mathematical statistics, this position is the main one and is considered as an axiomatic (*Park*, 2018; *Panchev*, 2013;). It is quite possible that in those branches of modern science where statistics of a complex random variable are used, this is the case. As the review of published works in this area shows, the complex random variable is mainly used in the signal theory

(*Schreier Peter J., Scharf Louis, 20*10; *Steven M.Kay*, 2010; *Tuelay Adili, Peter J.Schreier, Louis L. Scharf*, 2011 etc.) and there the independence of two signals of each other is quite a natural phenomenon. And since modern scientists mainly consider this particular case, we cannot ignore it, although it was previously shown that such cases are meaningless for the economy.

Since the real and imaginary parts of such a complex variable do not depend on each other, then all statistical characteristics of a complex random variable do not depend on each other.

Then the variance of the real part of the complex random variable will be equal to:

$$\sigma_r^2 = M\left|(y_r - m_{y_r})^2\right|, \quad (1.3.3)$$

and the variance of its imaginary part is:

$$\sigma_i^2 = M\left|(y_i - m_{y_i})^2\right|. \quad (1.3.4)$$

The total variance of a complex random variable with independent real and imaginary parts will be equal to the sum of its real and imaginary parts variances:

$$\sigma_Y^2 = \sigma_r^2 + \sigma_i^2. \quad (1.3.5)$$

It should be noted that variance is an important characteristic that makes it possible to describe the probability distribution density. We will exclusively consider the normal probability distribution, since most often we have to deal with it in practice.

The Gauss formula for the real part of a complex number will look like this:

$$f(y_r) = \frac{1}{\sigma_r \sqrt{2\pi}} e^{-\frac{(y_r - m_{y_r})^2}{2\sigma_r^2}} \quad . \qquad (1.3.6)$$

Similarly, we can write down the probability distribution density formula for the imaginary part of a complex number:

$$f(y_i) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{(y_i - m_{y_i})^2}{2\sigma_i^2}} \quad . \qquad (1.3.7)$$

Then, due to the independence of the real and imaginary parts of a complex random variable from each other, its distribution density will be equal to the product of the distribution densities of the real and imaginary parts:

$$f(Y) = f(y_r)\, f(y_i) = \frac{1}{2\pi\sigma_r\sigma_i} e^{-\frac{(y_r - m_{y_r})^2 \sigma_i^2 + (y_i - m_{yi})^2 \sigma_r^2}{2\sigma_r^2\sigma_i^2}} \quad . \qquad (1.3.8)$$

The form of this distribution is shown in Fig. 1.1, where the axes of the horizontal plane are the real and imaginary parts of a complex random variable (complex plane), and its distribution density is plotted horizontally.

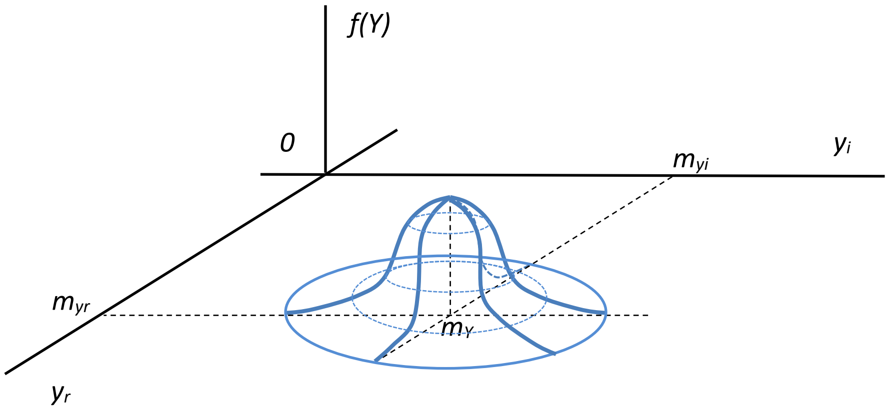

Figure 1.1. Mathematical expectation of a complex random variable with independent real and imaginary parts

All points on the complex plane have a different probability of occurrence – the further away from the mathematical expectation of mY, the less likely they are to appear on the complex plane

All points lying on a straight line with the coordinate *mi* have a different probability of occurrence, and the maximum probability of a random complex variable occurrence falls on the mathematical expectation point *my.* Similarly, all points lying on a straight line with the coordinate *mr* have different probability of occurrence, but the maximum probability of occurrence of a random complex variable on this line also falls on the mathematical expectation point *my*.

It can be seen from the figure that the cross-section of the surface in Fig. 1.1 with planes parallel to the complex plane, that is, planes of equal probability density, gives different ellipses. In probability theory, these ellipses are called "scattering ellipses", the equation of which in our case is determined by the variance of each of the parts and their mathematical expectation:

$$\frac{(y_r - m_{y_r})^2}{\sigma_{y_r}^2} + \frac{(y_i - m_{y_i})^2}{\sigma_{y_i}^2} = const$$

(1.3.9)

Ellipses can be easily projected onto the complex plane (Fig. 1.2).

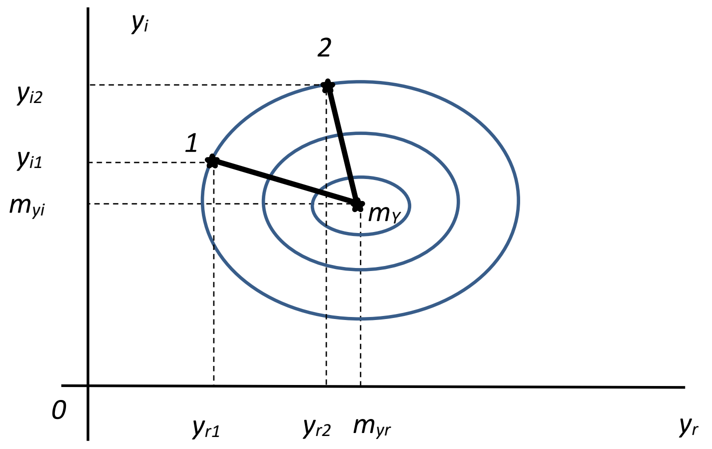

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Figure 1.2. Scattering ellipses for the situation of independence of the real and imaginary parts of a complex random variable.

In this figure, two points are plotted - 1 and 2, which characterize two different random complex numbers *(yr1; yi1*) and (*yr1; yi1*) . If we calculate the distances from them to the mathematical expectation, we get:

$$r_1 = \sqrt{(y_{r1} - m_{y_r})^2 + (y_{i1} - m_{y_i})^2} \text{ and } r_2 = \sqrt{(y_{r2} - m_{y_r})^2 + (y_{i2} - m_{y_i})^2} \qquad (1.3.10)$$

It is known that for the discrete case, the variance of a random variable *x* can be written as:

$$\sigma_x^2 = \sum_j p_j (x_j - m_x)^2 \quad .(1.3.11)$$

For a complex random variable (1.3.1), the form of the variance record will be as follows:

$$\sigma_Y^2 = \sum_j p_j [(y_{rj} - m_{y_r})^2 + (y_{ij} - m_{y_i})^2] \quad . \quad (1.3.12).$$

As it clearly follows from (1.3.12), the variance of a complex random variable with real and imaginary parts independent of each other will be the sum of the squared distances from the random variables lying on the complex plane to their mathematical expectation multiplied by the probabilities of these random variables' occurrence.

If the independent variances of the real and imaginary parts are equal to each other, then the ellipses of Fig. 2 turn into scattering circles.

The second case. The real and imaginary parts of a complex random variable are dependent on each other.

It would seem that in mathematical statistics it was quite logical to study all possible variants of a complex random variable properties. And if there is a variant of the real and imaginary parts of a complex random variable independence from each other, then there should be the second option - the variant of the real and imaginary parts of a complex random variable dependence on each other. Then all the options will be considered and the scientists will get full knowledge about the subject of research.

But it is just the variant of the real and imaginary parts of a complex random variable dependence on each other that the scientists did not consider in full. To all such cases and statistical characteristics, they, thanks to H. Harter and M. Lum`s example, add the prefix "pseudo" (Harter, 1955). And there are historical reasons for this.

Interest in statistical processing of observations of changes in a complex variable arose in the 50s-60s of the twentieth century. For the first time this problem was formulated by R. Wooding, who proposed an approach of a complex random variable representation from the standpoint of a normal distribution (Wooding, 1956). This approach was developed in their works by R. Arens (Arens, 1957) and I. Reed (Reed, 1962). A priori, these publications assumed the independence of the normally distributed real and imaginary parts of a complex random variable from each other, but this was not explicitly mentioned. In 1963, N. Goodman explicitly formulated this assumption (Goodman, 1963). Based on it, scientists further formulated the basic concepts and characteristics of a random normally distributed complex variable: mathematical expectation, moments (including the correlation moment), covariance, variance, etc. (Feller, 1966).

Modern researchers who use complex random variables in their scientific works always use the option when both parts of it are independent of each other (Tavares, 2006). There were, however, the first attempts to comprehend the situation when the real and imaginary parts depend on each other, but scientists considering this option immediately add "pseudo" - pseudo covariance or pseudo variance, etc. (Picinbono, 1997; Kammeyer, 2002; Soroush, 2010; Adali, 2010; Tuelay, 2011; Siu-Kui Au, 2017 etc.). And they do not go further than calculating pseudo moments, pseudo variances and pseudo covariances.

Therefore, it turned out that we have no ready-made solutions proposed by mathematical statistics for random complex variable whose real and imaginary parts depend on each other. We will have to deal with this issue on our own, checking with the level that modern mathematical statistics offers us in this matter.

Since random variables are considered, the relationship between them will be correlative. Denote by r_{ri} the coefficient of paired correlation between the real y_r and the imaginary y_i part of a complex random variable

The density of the normal distribution of two random interconnected quantities, as is known from the theory of probability and mathematical statistics, taking into account the notation we have adopted, will take the form:

$$f(y_r;y_i)=\frac{1}{2\pi\sigma_{y_r}\sigma_{y_i}\sqrt{1-r_{y_ry_i}^2}}e^{-\frac{1}{2(1-r_{y_ry_i}^2)}\left(\frac{(y_r-m_{y_r})^2}{\sigma_{y_r}^2}-2\frac{r_{y_ry_i}(y_r-m_{y_r})(y_i-m_{y_i})}{\sigma_{y_r}\sigma_{y_i}}+\frac{(y_i-m_{y_i})^2}{\sigma_{y_i}^2}\right)}. \quad (1.3.13)$$

One can make sure that when the pair correlation coefficient *ryryi* is equal to zero, the formula (1.3.13) turns into the formula (1.3.8).

This formula is used in modern mathematical statistics to describe the probabilistic characteristics of normally distributed random complex variables (Trampitsch, 2013, p. 40) .

The density of a complex random variable normal distribution with its interconnected parts has in three-dimensional space approximately the same form as shown in Fig. 1.1, but with a slight difference. As can be seen from Fig. 1.1, with the independence of the real and imaginary parts of a complex random variable, the three-dimensional model of the distribution density is symmetric with respect to the lines passing through the mathematical expectation point and parallel to the axes of the complex plane.

And in the case of the dependence of these real and imaginary parts of a complex random variable on each other, the model becomes asymmetric to these lines. It becomes symmetrical to the lines that are not parallel to the axes of the complex plane (Fig. 1.3). In this case, the scattering ellipses also change their position

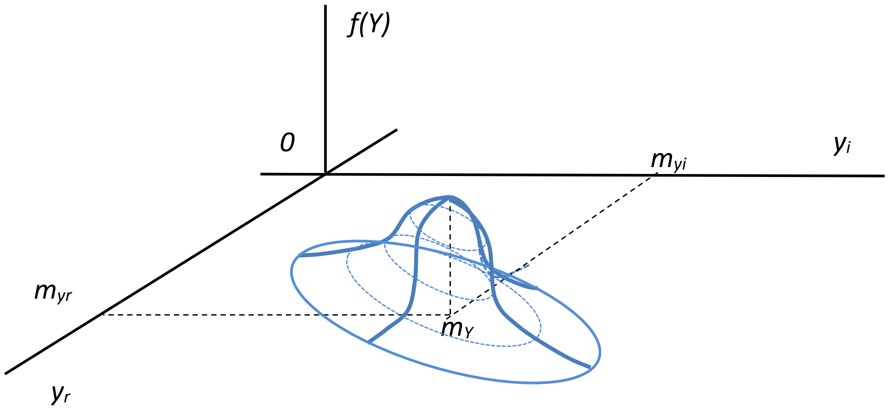

Figure 1.3. Mathematical expectation and probability distribution density with dependent on each other parts of a complex random variable

It is more convenient to consider not a three-dimensional figure in space, but ellipses of scattering.

They, for the case of the existing dependence between the real and imaginary parts, will have the following form

$$\frac{(y_r - m_{y_r})^2}{\sigma_{y_r}^2} - 2\frac{r_{y_r y_i}(y_r - m_{y_r})(y_i - m_{y_i})}{\sigma_{y_r}\sigma_{y_i}} + \frac{(y_i - m_{y_i})^2}{\sigma_{y_i}^2} = const \tag{1.3.14}$$

Figure 1.4 shows one of these scattering ellipses.

And it is characteristic for it that the distances from the points lying on the ellipse to the mathematical expectation m_Y are equal

$$r_1 = \sqrt{(y_{r1} - m_{y_r})^2 + (y_{i1} - m_{y_i})^2} \quad \text{и} \quad r_2 = \sqrt{(y_{r2} - m_{y_r})^2 + (y_{i2} - m_{y_i})^2} \tag{1.3.15}$$

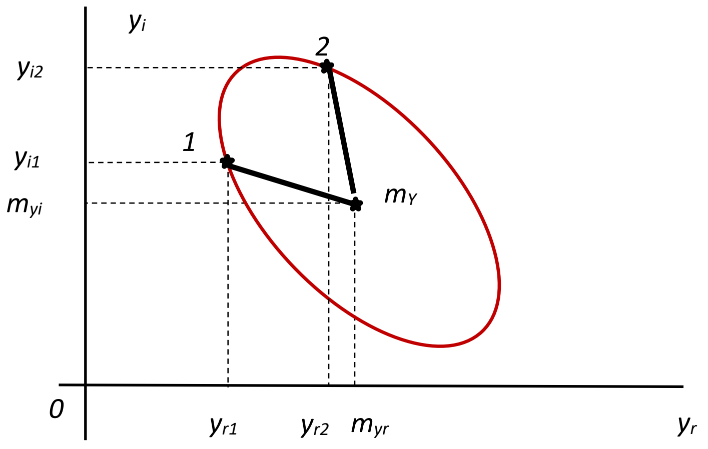

Figure 1.4. The scattering ellipse on the complex plane with the relationship between the real and imaginary parts of a complex random variable

And the variance is respectively equal to:

$$\sigma^2_{Y_{ri}} = \sum_j p_j^{ri}[(y_{rj} - m_{y_r})^2 + (y_{ij} - m_{y_i})^2] \tag{1.3.16}$$

Here- p_j^{ri} is the probability of a complex random variable occurrence corresponding to (1.3.13).

Since the probabilities in the case of a) the independence of the real and imaginary parts and in case of b) their dependences on each other are of different nature and are differently calculated, this means that the variance in the latter case cannot be calculated as in the case of the parts of a complex random variable independence from each other (1.3.12), that is:

$$\sigma^2_Y = \sum_j p_j[(y_{rj} - m_{y_r})^2 + (y_{ij} - m_{y_i})^2] \neq \sigma^2_{Y_{ri}} = \sum_j p_j^{ri}[(y_{rj} - m_{y_r})^2 + (y_{ij} - m_{y_i})^2]$$

Let us show it.

In Fig. 5, two scattering ellipses corresponding to the same value of the probability distribution density are plotted on the complex plane. The probability of points appearing on the lines of these ellipses is the same. But the blue scattering ellipse corresponds to the situation of the real and imaginary parts of a complex random variable independence from each other, and the red ellipse corresponds to the second variant, when the real and imaginary parts of a complex random variable depend on each other.

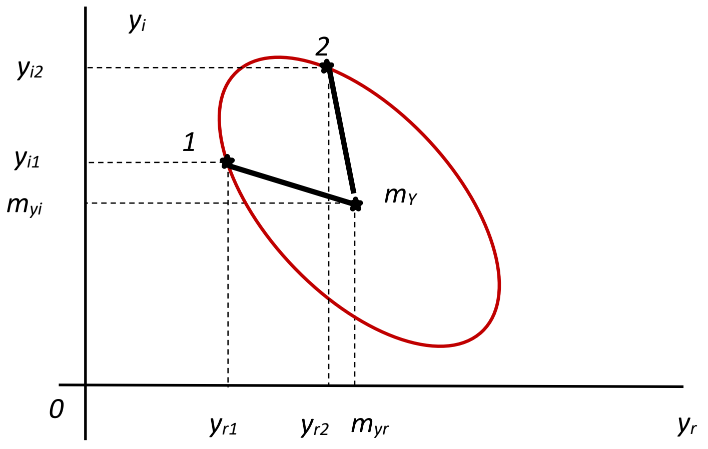

Figure 1.5. Scattering ellipses on the complex plane with the interrelation between the real and imaginary parts of a complex random variable (red) and in the case of their independence (blue)

Point 1 lies both on the line of the blue scattering ellipse and on the line of the red scattering ellipse. The probability of this point occurrence is the same both in the case of the real and imaginary parts dependence on each other and in the case of their independence from each other. But point 2 lies only on the line of the red ellipse and is above the blue scattering ellipse. This means that the probability p of this point occurrence in the case of the real and imaginary parts of a complex random variable independence from each other is less than the probability p_{ri} of this point occurrence in the case of the components of a complex random variable dependence on each other:

$$p < p_{ri}.$$

Then:

$$p[(y_{rj}-m_{y_r})^2+(y_{ij}-m_{y_i})^2] < p^{ri}[(y_{rj}-m_{y_r})^2+(y_{ij}-m_{y_i})^2]. \qquad (1.3.18)$$

Since the variance of a complex random variable with the mutual dependence of the real and imaginary parts is not a simple sum of the variances of the real and imaginary parts, its properties should be studied in more detail.