## 1.7. Nonlinear correlations or lack of correlation

Models of complex variables describe the processes under study differently than models of real variables. Therefore, the complex coefficient of pair correlation should also be expected to have new properties that are not inherent in the coefficient of paired correlation of real variables.

At the very beginning of these new properties` study, let us pay attention to the situation when the real part of the complex coefficient of pair correlation is zero, and its imaginary part is equal in modulus to one:

$$r_{XY} = \pm(0+i). \quad (1.7.1)$$

This case, as it is easy to see, is the opposite of the just considered case of linear correlation(1.6.3).

For (1.7.1), the following condition must be met:

$$ab = R_a R_b e^{i(\alpha+\beta)} = R_a R_b[\cos(\alpha+\beta) + i\cos(\alpha+\beta)] = 0 + i. \quad (1.7.2)$$

This in turn means that

$$R_a R_b = 1, \quad \alpha+\beta = (2k-1)\pi, \quad k = 1,2,3,... \quad (1.7.3)$$

What does the obtained equality show, for example, when *k=1*?

It shows that if, for a proportionality complex coefficient of the complex argument regression to a complex result equal to

$$a_0 + ia_1 = 1 + i0,$$

the least-squares method (LSM) , applied to the inverse dependence - of the argument on the result - should give such estimates of the complex coefficient:

$$b_0 + ib_1 = -1 + i0$$

Only in this case the condition (1.7.3) will be met.

The same condition will be met if, when estimating the LSM on the set of values of complex random variables, the coefficient of the result dependence on the argument

$$a_0 + ia_1 = 1 + i$$

will correspond to the inverse linear dependence coefficient equal to

$$b_0 + ib_1 = -1 - i$$

that is, a vector opposite in the complex plane towards the first one.

Now we can understand in which case the real part of the complex coefficient of pair correlation will be zero, and its imaginary part modulo will be equal to one.

When finding the regression of a complex argument to a complex result

$$y_r + iy_i = (a_0 + ia_1)(x_r + ix_i)$$

the proportionality coefficient $a_0 + ia_1$ found by the LSM will simulate some linear sequence $\hat{y}_r + i\hat{y}_i$ as close as possible to the original series of a random variable.

When using LSM to find on the same data the inverse dependence of the complex argument on the complex result:

$$x_r + ix_i = (b_0 + ib_1)(y_r + iy_i),$$

LSM will give such a complex coefficient $b_0 + ib_1$ that its use for regression

$$y_r + iy_i = \frac{x_r + ix_i}{b_0 + ib_1}$$

will simulate a series of points $\hat{y}'_r + i\hat{y}'_i$ rotated relative to the original series $\hat{y}_r + i\hat{y}_i$ by an angle of *π*, that is, changing in the opposite direction.

This is possible only in case if there is a complete absence of linear dependence and in general any dependence between two random complex variables, that is,

- *when the two analyzed random complex variables are absolutely independent of each other.*

Thus, we have shown what the values of the complex pair correlation coefficient mean:

- if the complex pair correlation coefficient is ±(1+i0), then this indicates the presence of a linear functional relationship between the variables;

- if the complex pair correlation coefficient is ±(0+i), then this indicates that there is complete independence of two complex random variables from each other.

However, what has been proved does not at all allow us to assert that the complex pair correlation coefficient lies in the range from ±(*0+i*) to ±(*1+i0*) and in this interval it characterizes a different degree of approximation to a linear complex-valued dependence. The second statement is generally true – the closer the calculated values of the complex pair correlation coefficient are to ±(*1+i0*), the closer the relationship between them is to a linear form. But the first statement about the limits of the change in the complex pair correlation coefficient is not true. The behavior of the complex pair correlation coefficient is much more complicated than of its real counterpart.

In order to understand how the complex pair correlation coefficient can change its values, we will reveal the numerator of this coefficient:

$$\sum (y_{rt} + iy_{it})(x_{rt} + ix_{it}) = \sum (y_{rt}x_{rt}) - \sum y_{it}x_{it} + i(\sum y_{it}x_{rt} + \sum y_{rt}x_{it})$$

. (1.7.4)

Let's write it down through covariances (after all, all the original variables are centered relative to their arithmetic averages). Then we get:

$$\sum(y_{rt}+iy_{it})(x_{rt}+ix_{it}) = \operatorname{cov}(y_{rt}x_{rt}) - \operatorname{cov}(x_{it}y_{it}) + i[\operatorname{cov}(y_{it}x_{rt}) + \operatorname{cov}(y_{rt}x_{it})] \quad (1.7.5)$$

Why are covariances important to us? Because it is known from mathematical statistics that the covariance of two independent random variables is zero.

The denominator of the complex pair correlation coefficient can be written in terms of complex variances, the properties of which have already been well known to us (1.5.2). Then the complex pair correlation coefficient in such a more detailed and convenient for an analysis form will be written as follows:

$$r_{c\,XY} = \frac{\operatorname{cov}(y_{rt}x_{rt}) - \operatorname{cov}(x_{it}y_{it}) + i[\operatorname{cov}(y_{it}x_{rt}) + \operatorname{cov}(y_{rt}x_{it})]}{\sqrt{\sigma^2_{x_{rt}} - \sigma^2_{x_{it}} + i2\operatorname{cov}(x_{rt}x_{it})}\sqrt{\sigma^2_{y_{rt}} - \sigma^2_{y_{it}} + i2\operatorname{cov}(y_{rt}y_{it})}}. \quad (1.7.6)$$

This very form of the correlation coefficient recording already makes it possible to judge substantively its properties.

When the variances of the imaginary and real parts of each of the original variables are equal to each other, that is:

$$\sigma^2_{x_{rt}} - \sigma^2_{x_{it}} = 0, \quad \sigma^2_{y_{rt}} - \sigma^2_{y_{it}} = 0, \quad (1.7.7)$$

then we get for the denominator:

$$i2\sqrt{\operatorname{cov}(x_{rt}x_{it})\operatorname{cov}(y_{rt}y_{it})}. \quad (1.7.8)$$

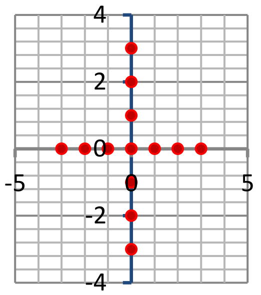
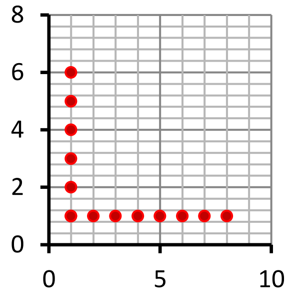
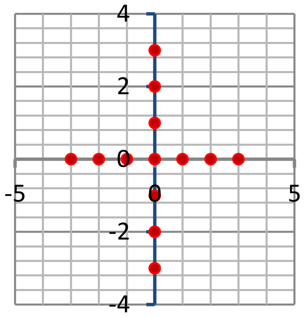
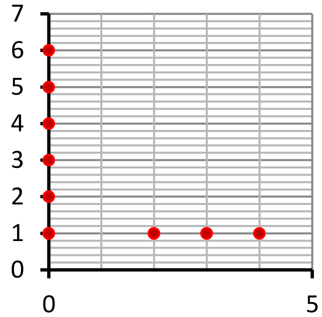
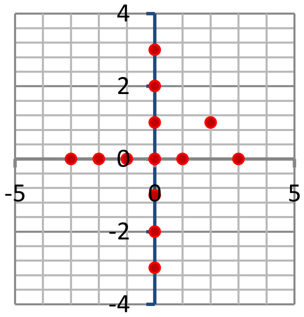
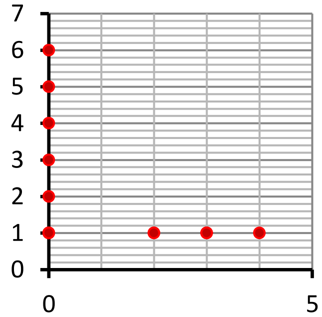
If it still happens that the real and imaginary parts of complex variables are independent of each other, then the covariances in the radicand (1.7.8) will be equal to zero. It should be noted that for sample values they will be rarely equal to zero, just like (1.7.7) will be strictly equal to zero for sample values extremely rarely. In the overwhelming majority of cases, both (1.7.7) and (1.7.8) will be close to zero, but not strictly equal to it.

In the real and imaginary parts of the numerator (1.7.6) there are cross covariances, which in the sample case can be even less often equal to zero. Therefore, dividing the numerator by a value close to zero values will result in a situation where the complex coefficient of pair correlation can have high values of both real and imaginary parts, and these values can significantly exceed one in modulus.

Examples of such cases are given in Table 1.1

Table 1.1.

*Exceptional cases of the complex coefficient of pair correlation values*

| № | Graph of the complex variable X | Variance of the complex variable X | Graph of the complex variable Y | Variance of the complex variable Y | Complex coefficient of pair correlation |
|---|---|---|---|---|---|
| 1. |   | *0-i15,0* |   | *45,1-i59,9* | *33651969-i11231561* |
| 2. |   | *0-i15,0* |   | *-8,6-i36,4* | *-5544718-i643532* |
| 3. |   | *-0,928+i4,0* |   | *-8,6-i36,4* | *-0,316-i0,259* |

In the real practice of econometric models' construction, the situation in which such conditions develop is unlikely, if only because the real and imaginary parts of such random complex variables are dependent on each other - otherwise their formation in the economy is meaningless! Therefore, even if the variances of the real and imaginary parts of each of the considered complex random variables are equal to each other (an extremely rare case for sample values), the denominator (1.7.6) will still be far from zero.

Knowing this, it is possible to interpret the values of the real part of the complex coefficient of pair correlation in this way: there can be a linear correlation between the two complex random variables if the condition is met:

$$\left|1-\operatorname{Re}(r_{cXY})\right| \le 0,2 \quad (1.7.9)$$

In all other cases, the linear relationship will not be so close and, most likely, the researcher needs to look for another form of dependence between two random complex variables.